ON THE REDUCTION OF THE EQUATIONS OF MOTION OF A GYROHORIZONCOMPASS

(O PRIVODIMOSTI URAVNENII DVIZHENIIA GIROGORIZONTKOMPASA)

PMM Vol.25, No.5, 1961, pp. 801-805

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(Received June 3, 1961)

It is shown in this paper that the equations of perturbed motion of a gyrohorizoncompass, given in [1], can be reduced to a system with constant coefficients.

A rigorous analytic justification of the passage to the simplified equations of Geckeler is presented. The influence of an external periodic force is also considered.

1. The equations of perturbed motion of a spatial gyrohorizoncompass of Geckeler-Anschütz [1] are of the form

$$\frac{d}{dt}\frac{V\alpha}{\sqrt{gR}} - \nu\beta = \Omega\frac{2B\sin\varepsilon^\circ}{ml\sqrt{gR}}\delta, \qquad \frac{d\beta}{dt} + \nu\frac{V\alpha}{\sqrt{gR}} = \Omega\gamma$$
$$\frac{d\gamma}{dt} + \nu\frac{2B\sin\varepsilon^\circ}{ml\sqrt{gR}}\delta = -\Omega\beta, \qquad \frac{d}{dt}\left(\frac{2B\sin\varepsilon^\circ}{ml\sqrt{gR}}\delta\right) - \nu\gamma = -\Omega\frac{V\alpha}{\sqrt{gR}} \tag{1.1}$$

Here

$$V = \sqrt{(Ru\cos\varphi + v_E)^2 + v_N{}^2} \tag{1.2}$$
$$\Omega = u\sin\varphi + \frac{v_E}{R}\tan\varphi + \frac{d\alpha^*}{dt} \qquad \left(\alpha^* = \frac{v_N}{Ru\cos\varphi + v_E}\right)$$

It is assumed that the ship is maneuvering arbitrarily along a fixed latitude ϕ and in system (1.1) new variables

$$\alpha = \frac{Ru\cos\varphi}{V}x_1, \qquad \delta = \frac{\sin\varphi}{\sin\varepsilon^\circ}x_4 \tag{1.3}$$

are introduced, such that ϵ° satisfies the condition

$$\varepsilon^\circ = \cos^{-1} \frac{mlV}{2B} \qquad (1.4)$$

Further, β and γ are also expressed through x_2 and x_3, respectively. We obtain the following system:

$$\dot{x}_1 = \frac{\nu^2}{u\cos\varphi} x_2 + \lambda\Omega\tan\varphi x_4, \qquad \dot{x}_3 = -\Omega x_2 - \frac{\nu^2 2B\sin\varphi}{Pl} x_4 \qquad (1.5)$$

$$\dot{x}_2 = -u\cos\varphi x_1 + \Omega x_3, \qquad \dot{x}_4 = -\frac{1}{\lambda}\Omega\cot\varphi x_1 + \frac{Pl}{2B\sin\varphi} x_3$$

where

$$\lambda = \frac{2Bg}{PlRu} \qquad (1.6)$$

If in system (1.5) those terms are neglected which contain the angular velocity Ω as a factor, then it uncouples into two independent systems of the form

$$\dot{x}_1 = \frac{\nu^2}{u\cos\varphi} x_2, \quad \dot{x}_2 = -u\cos\varphi x_1, \quad \dot{x}_3 = -\frac{\nu^2 2B\sin\varphi}{Pl} x_4, \quad \dot{x}_4 = \frac{Pl}{2B\sin\varphi} x_3 \qquad (1.7)$$

which determine the harmonic undamped oscillations of the compass with an angular frequency ν.

The simplified equations (1.7), apparently obtained first by Geckeler [2], form the basis of the majority of studies and texts on the theory of the gyrohorizoncompass.

2. We pass in system (1.5) to new variables with the aid of the non-singular substitution of the form

$$\begin{aligned}
\xi_1 &= x_1\cos\theta - \frac{\nu}{u\cos\varphi} x_2\cos\theta + \frac{\nu}{u\cos\varphi} x_3\sin\theta - \lambda\tan\varphi x_4\sin\theta \\
\xi_2 &= \frac{u\cos\varphi}{\nu} x_1\cos\theta + x_2\cos\theta - x_3\sin\theta - \frac{\nu 2B\sin\varphi}{Pl} x_4\sin\theta \\
\xi_3 &= \frac{u\cos\varphi}{\nu} x_1\sin\theta + x_2\sin\theta + x_3\cos\theta + \frac{\nu 2B\sin\varphi}{Pl} x_4\cos\theta \qquad (2.1) \\
\xi_4 &= \frac{1}{\lambda}\cot\varphi x_1\sin\theta - \frac{Pl}{\nu 2B\sin\varphi} x_2\sin\theta - \frac{Pl}{\nu 2B\sin\varphi} x_3\cos\theta + x_4\cos\theta
\end{aligned}$$

where

$$\theta(t) = \int_0^t \Omega(\tau)\,d\tau \qquad (2.2)$$

As a result, we are led to a system of equations with respect to ξ_k, which uncouples into two independent systems with constant coefficients and possesses the same structure as the system (1.7), namely

$$\dot{\xi}_1 = \frac{\nu^2}{u\cos\varphi}\xi_2, \quad \dot{\xi}_2 = -u\cos\varphi\xi_1, \quad \dot{\xi}_3 = -\frac{\nu^2 2B\sin\varphi}{Pl}\xi_4, \quad \dot{\xi}_4 = \frac{Pl}{2B\sin\varphi}\xi_3 \qquad (2.3)$$

The system (1.5) is thus reducible to the system of Geckeler (1.7).

We also give the formulas for the inverse transformation from variables ξ_k to the variables x_k, which will be used in the sequel. We have

$$x_1 = \frac{1}{2}\left(\xi_1\cos\theta + \frac{\nu}{u\cos\varphi}\xi_2\cos\theta + \frac{\nu}{u\cos\varphi}\xi_3\sin\theta + \lambda\tan\varphi\xi_4\sin\theta\right)$$

$$x_2 = \frac{1}{2}\left(-\frac{u\cos\varphi}{\nu}\xi_1\cos\theta + \xi_2\cos\theta + \xi_3\sin\theta - \frac{\nu 2B\sin\varphi}{Pl}\xi_4\sin\theta\right)$$

$$x_3 = \frac{1}{2}\left(\frac{u\cos\varphi}{\nu}\xi_1\sin\theta - \xi_2\sin\theta + \xi_3\cos\theta - \frac{\nu 2B\sin\varphi}{Pl}\xi_4\cos\theta\right) \qquad (2.4)$$

$$x_4 = \frac{1}{2}\left(-\frac{1}{\lambda}\cot\varphi\,\xi_1\sin\theta - \frac{Pl}{\nu 2B\sin\varphi}\xi_2\sin\theta + \frac{Pl}{\nu 2B\sin\varphi}\xi_3\cos\theta + \xi_4\cos\theta\right.$$

3. We assume that the ship performs sequential circulations with constant velocity v and a circular frequency on a given latitude ϕ, beginning, for example, with the course due north.

Then, as is shown in [2], we may assume

$$\Omega \approx -\mu\omega\sin\omega t \qquad \left(\mu = \frac{v}{Ru\cos\varphi}\right) \qquad (3.1)$$

With this assumption the system (1.5) will be

$$(3.2)$$

$$\dot{x}_1 = \frac{\nu^2}{u\cos\varphi}x_2 - \lambda\mu\omega\tan\varphi\sin\omega t x_4, \quad \dot{x}_3 = \mu\omega\sin\omega t x_2 - \frac{\nu^2 2B\sin\varphi}{Pl}x_4$$

$$\dot{x}_2 = -u\cos\varphi x_1 - \mu\omega\sin\omega t x_3, \qquad \dot{x}_4 = \frac{1}{\lambda}\mu\omega\cot\varphi\sin\omega t x_1 + \frac{Pl}{2B\sin\varphi}x_3$$

The system associated with (3.2) is of the form

$$(3.3)$$

$$\dot{y}_1 = u\cos\varphi y_2 - \frac{1}{\lambda}\mu\omega\cot\varphi\sin\omega t y_4, \quad \dot{y}_3 = \mu\omega\sin\omega t y_2 - \frac{Pl}{2B\sin\varphi\nu}y_4$$

$$\dot{y}_2 = -\frac{\nu^2}{u\cos\varphi}y_1 - \mu\omega\sin\omega t y_3, \qquad \dot{y}_4 = \lambda\mu\omega\tan\varphi\sin\omega t y_1 + \frac{\nu^2 2B\sin\varphi}{Pl}y_3$$

By means of variables [1]

$$(3.4)$$

$$w_1(t) = \frac{\nu}{u\cos\varphi}y_1 + iy_2, \qquad w_2(t) = y_3 - i\frac{Pl}{2B\sin\varphi\nu}y_4 \qquad (i = \sqrt{-1})$$

we transform the system (3.3) into an easily integrable system of two equations of first order.

If $y_i(t)$ is any solution of system (3.3), then, as is known [4], the expression $y_1x_1 + y_2x_2 + y_3x_3 + y_4x_4$ will be the first integral of the system (3.2). Using Formulas (3.4) and satisfying in the solutions of the system (3.3) the initial conditions

$$y_{ik}(0) = \delta_{jk} = \begin{cases} 1 & (j = k) \\ 0 & (j \neq k) \end{cases}$$

we construct real expressions for the four independent first integrals of system (3.2). As a result we obtain

$$x_1 \cos \nu t \cos \theta - \frac{\nu}{u \cos \varphi} x_2 \sin \nu t \cos \theta + \frac{\nu}{u \cos \varphi} x_3 \sin \nu t \sin \theta - \lambda \tan \varphi\, x_4 \cos \nu t \sin \theta = C_1$$

$$\frac{u \cos \varphi}{\nu} x_1 \sin \nu t \cos \theta + x_2 \cos \nu t \cos \theta - x_3 \cos \nu t \sin \theta - \frac{\nu 2B \sin \varphi}{Pl} x_4 \sin \nu t \sin \theta = C_2$$

$$\frac{u \cos \varphi}{\nu} x_1 \sin \nu t \sin \theta + x_2 \cos \nu t \sin \theta + x_3 \cos \nu t \cos \theta + \frac{\nu 2B \sin \varphi}{Pl} x_4 \sin \nu t \cos \theta = C_3$$

$$\frac{1}{\lambda} \cot \varphi\, x_1 \cos \nu t \sin \theta - \frac{Pl}{\nu 2B \sin \varphi} x_2 \sin \nu t \sin \theta - \frac{Pl}{\nu 2B \sin \varphi} x_3 \sin \nu t \cos \theta + x_4 \cos \nu t \cos \theta = C_4$$

Here, in accordance with (2.2) and (3.1), in the case of circulation it must be assumed

$$\theta(t) = \mu(\cos \omega t - 1) \tag{3.5}$$

It already becomes clear that, as a linear substitution with periodic coefficients, which reduces the system (3.2) to a system with constant coefficients (2.3), Expressions (2.1) should be taken, where $\theta(t)$ satisfies Formula (3.5).

The roots of the characteristic equations of the transformed system (2.3) are, as is known, the characteristic exponents of system (3.2). Designating the latter by κ_s ($s = 1, 2, 3, 4$) we obtain

$$\varkappa_{1,2} = \pm \nu i, \qquad \varkappa_{3,4} = \pm \nu i \tag{3.6}$$

4. Let us apply the theory presented to a study of the influence of an external periodic disturbance.

Let us consider the nonhomogeneous system

$$\dot{x}_1 = \frac{\nu^2}{u \cos \varphi} x_2 - \lambda \mu \omega \tan \varphi \sin \omega t x_4 \tag{4.1}$$

$$\dot{x}_2 = -u\cos\varphi x_1 - \mu\omega\sin\omega t x_3 + F(t)$$

$$\dot{x}_3 = \mu\omega\sin\omega t x_2 - \frac{\nu^2 B\sin\varphi}{Pl} x_4$$

$$\dot{x}_4 = \frac{1}{\lambda}\mu\omega\cot\varphi\sin\omega t x_1 + \frac{Pl}{2B\sin\varphi} x_3$$

and let

$$F(t) = a\cos\omega t \qquad (a > 0) \tag{4.2}$$

Passing, in accordance with (2.1), to the variables ξ_k, we obtain two independent systems of the form

$$\begin{aligned}\dot{\xi}_1 &= \frac{\nu^2}{u\cos\varphi}\xi_2 - \frac{\nu}{u\cos\varphi}F(t)\cos\theta(t), \quad & \dot{\xi}_3 &= -\frac{\nu^2 2B\sin\varphi}{Pl}\xi_4 + F(t)\sin\theta(t)\\ \dot{\xi}_2 &= -u\cos\varphi\,\xi_1 + F(t)\cos\theta(t), \quad & \dot{\xi}_4 &= \frac{Pl}{2B\sin\varphi}\xi_3 - \frac{Pl}{\nu 2B\sin\varphi}F(t)\sin\theta(t)\end{aligned} \tag{4.3}$$

where θ, as before, satisfies Formula (3.5).

We will assume, further, that $\mu \ll 1/2$; we then may set $\sin\theta \approx \theta$, $\cos\theta \approx 1$, and the system (4.3) will be

$$\begin{aligned}\dot{\xi}_1 &= \frac{\nu^2}{u\cos\varphi}\xi_2 - \frac{\nu}{u\cos\varphi}F(t), \quad & \dot{\xi}_3 &= -\frac{\nu^2 2B\sin\varphi}{Pl}\xi_4 + F(t)\theta(t)\\ \dot{\xi}_2 &= -u\cos\varphi\xi_1 + F(t), \quad & \dot{\xi}_4 &= \frac{Pl}{2B\sin\varphi}\xi_3 - \frac{Pl}{\nu 2B\sin\varphi}F(t)\theta(t)\end{aligned} \tag{4.4}$$

It is important to note that in the case considered the term $F(t)\theta(t)$ has a constant component.

Indeed, in accordance with (3.5) and (4.2)

$$F(t)\theta(t) = \mu a\cos\omega t(\cos\omega t - 1) = \frac{\mu a}{2} + \frac{\mu a}{2}\cos 2\omega t - \mu a\cos\omega t \tag{4.5}$$

This constant component, expressed by the first term in Formula (4.5), is of considerable influence on the reading of the gyrohorizoncompass.

We proceed now to the integration of system (4.4). We have from the first two equations of this system, taking into account Formula (4.2) and the initial conditions $\xi_1(0) = 0$, $\xi_2(0) = 0$, the following solutions:

$$\begin{aligned}\xi_1 &= \frac{\nu^2 a}{(\nu^2 - \omega^2)u\cos\varphi}\left(\cos\omega t - \cos\nu t + \frac{\omega}{\nu}\sin\omega t - \sin\nu t\right)\\ \xi_2 &= \frac{\nu a}{\nu^2 - \omega^2}\left(\cos\omega t - \cos\nu t + \sin\nu t - \frac{\omega}{\nu}\sin\omega t\right)\end{aligned} \qquad (\omega \neq \nu) \tag{4.6}$$

Assuming $\omega >> \nu$, we obtain from this approximate, but in many cases sufficiently accurate expressions

$$\xi_1 = \frac{\nu^2 a}{\omega^2 u \cos\varphi}\left(\cos\nu t - \cos\omega t - \frac{\omega}{\nu}\sin\omega t + \sin\nu t\right)$$
$$\xi_2 = \frac{\nu a}{\omega^2}\left(\cos\nu t - \cos\omega t + \frac{\omega}{\nu}\sin\omega t - \sin\nu t\right) \tag{4.7}$$

Further, from the remaining two equations of system (4.4), and taking into account (4.2), (4.5) and the initial conditions $\xi_3(0) = 0$, $\xi_4(0) = 0$, we obtain

$$\xi_3 = \mu a\left(\frac{\nu}{\nu^2-\omega^2} - \frac{1}{2}\frac{\nu}{\nu^2-4\omega^2} - \frac{1}{2\nu}\right)\cos\nu t + \frac{\mu a\omega^2}{\nu}\left(\frac{2}{\nu^2-4\omega^2} - \frac{1}{\nu^2-\omega^2}\right)\sin\nu t + $$
$$+ \frac{1}{2}\frac{\mu a}{\nu} + \frac{\mu\nu a}{\nu^2-4\omega^2}\left(\frac{1}{2}\cos 2\omega t - \sin 2\omega t\right) - \frac{\mu\nu a}{\nu^2-\omega^2}\left(\cos\omega t - \sin\omega t\right)$$
$$\xi_4 = \frac{Pl}{\nu^2 2B\sin\varphi}\left[\frac{a\mu}{2} + \frac{a\mu}{2}\cos 2\omega t - a\mu\cos\omega t + \right. \tag{4.8}$$
$$+ \mu a\nu\left(\frac{\nu}{\nu^2-\omega^2} - \frac{1}{2}\frac{\nu}{\nu^2-4\omega^2} - \frac{1}{2\nu}\right)\sin\nu t - \mu a\omega^2\left(\frac{2}{\nu^2-4\omega^2} - \frac{1}{\nu^2-\omega^2}\right)\cos\nu t + $$
$$\left. + \frac{\mu a\nu\omega}{\nu^2-4\omega^2}\sin 2\omega t - \frac{\mu\nu a\omega}{\nu^2-\omega^2}\sin\omega t + \frac{2\mu a\omega^2}{\nu^2-4\omega^2}\cos 2\omega t - \frac{\mu a\omega^2}{\nu^2-\omega^2}\cos\omega t\right]$$

Returning to Formula (4.6), we have

$$\xi_1 + \frac{\nu}{u\cos\varphi}\xi_2 = \frac{2\nu^2 a}{(\nu^2-\omega^2)\,u\cos\varphi}(\cos\omega t - \cos\nu t) \tag{4.9}$$

Neglecting ν^2 as compared to ω^2, we obtain from here

$$\xi_1 + \frac{\nu}{u\cos\varphi}\xi_2 = \frac{2\nu^2 a}{\omega^2 u\cos\varphi}(\cos\nu t - \cos\omega t) \tag{4.10}$$

Taking account of this simplification we also have

$$-\frac{u\cos\varphi}{\nu}\xi_1 + \xi_2 = \frac{2\nu a}{\omega^2}\left(\frac{\omega}{\nu}\sin\omega t - \sin\nu t\right) \tag{4.11}$$

$$\frac{\nu}{u\cos\varphi}\xi_3 + \lambda\tan\varphi\xi_4 = \frac{\mu a}{u\cos\varphi}(1 - \cos\nu t),\ \xi_3 - \frac{\nu\,2B\sin\varphi}{Pl}\xi_4 = \frac{\mu a}{\nu}\sin\nu t$$

From the formulas for inverse transformation (2.4), where in accordance with what has been said one should set $\sin\theta = \theta$, $\cos\theta = 1$, we have

$$x_1 = \frac{1}{2}\left[\frac{2\nu^2 a}{\omega^2 u\cos\varphi}(\cos\nu t - \cos\omega t) - \frac{\mu^2 a}{u\cos\varphi}(1 - \cos\nu t)(1 - \cos\omega t)\right]$$
$$x_2 = \frac{1}{2}\left[\frac{2\nu a}{\omega^2}\left(\frac{\omega}{\nu}\sin\omega t - \sin\nu t\right) - \frac{\mu^2 a}{\nu}\sin\nu t\,(1 - \cos\omega t)\right] \tag{4.12}$$

In these expressions the most important will be the last terms which are due to the presence of the constant component in Expressions (4.5). Designating them by Δx_1 and Δx_2 respectively, we have

$$\Delta x_1 = -\frac{1}{2}\frac{\mu^2 a}{u\cos\varphi}(1-\cos\nu t)(1-\cos\omega t)$$
$$\Delta x_2 = -\frac{1}{2}\frac{\mu^2 a}{\nu}\sin\nu t\,(1-\cos\omega t) \qquad (4.13)$$

If the periodic external force is acting during a short interval of time $(0, t^*)$ which is small as compared to the period of M. Schuler, then for $0 \leqslant t \leqslant t^*$ we may assume $\cos\nu t \approx 1$, $\sin\nu t \approx \nu t$; we then have

$$\Delta x_1 = 0, \qquad \Delta x_2 = -\tfrac{1}{2}\mu^2 a\,(1-\cos\omega t)\,t \qquad (4.14)$$

The maximum possible deviations will be

$$|\Delta x_{1m}| = 2\frac{\mu^2 a}{u\cos\varphi}, \qquad |\Delta x_{2m}| = \frac{\mu^2 a}{\nu} \qquad (4.15)$$

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Translated by G.H.